NASA TECHNICAL MEMORANDUM

47

Report No. 53934

COORDINATE SYSTEMS IN RELATIVISTIC FREQUENCY SHIFTS OF ATOMIC CLOCKS

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September 16, 1969

NASA

George C. Marshall Space Flight Center
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IN-SSL-N-67-14 November 6, 1967

This Internal Note was changed to NASA TM X-53934 on Sept. 16, 1969

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ABSTRACT

The theoretical possibility of eliminating the effects of Doppler, special relativity, and electromagnetic propagation through the atmosphere in making measurements of frequency shifts of two identical atomic oscillators is described. One of the oscillators is considered to be located on the surface of the Earth and the other one in synchronous orbit almost directly above the Earth-based oscillator. The frequency shifts measured will then arise principally from the pure gravitational field of the Earth and a rotational effect due to making the measurements in a uniformly rotating coordinate system. The technique is referred to as a double Doppler elimination procedure.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

I. INTRODUCTION

The purpose of this Note is to elucidate on the subject discussed in a recent Internal Note by Shelton, Edmonson, and Wills titled "Physics of the Clock Experiment." Detailed consideration will be given to the double Doppler elimination technique used in a uniformly rotating coordinate system to measure the absolute frequency shifts of atomic oscillators caused by a change in potential in an accelerated force field.

II. THE SPACE-FIXED COORDINATE SYSTEM

Consider a space-fixed coordinate system originating at the center of mass of the Earth with the z-axis coinciding with the Earth's axis of rotation (which is assumed to be space fixed). Let the x-axis be in the equatorial plane, and in particular, let it be positioned toward the first point of Aries, or vernal equinox (i. e., where the Sun crosses the equator in a northerly direction). Finally, let the y-axis be mutually perpendicular (in a right-handed sense) to the x and z axes and, of course, in the equatorial plane. Such a coordinate system remains nearly space fixed relative to the so called "fixed" stars located "semi-infinitely" away in configuration space.

Such a coordinate system will be used initially to formulate the physical hypothesis in the description of the experiment involving the atomic clocks. Then the description will be shifted to a uniformly rotating coordinate system rotating about the space

fixed z-axis. The speed of rotation will be considered identical to the speed of the Earth's rotation, which is assumed to be constant.

Operationally speaking, it will be assumed that the disadvantages of using such a space-fixed coordinate system will be well under the optimum operating efficiency of the atomic oscillators. Some of the well known disadvantages of using such a reference system are as follows:

1. The effect of geographical "wandering" of the poles causing small, but perceptible, variations in the latitudes of points on the Earth.

2. The lunisolar precession due to the pull of the Moon and Sun on the Earth's equatorial bulge causing the Earth's axis to describe a cone in space over a period of about 26,000 years.

3. A nutation or "nodding" of the Earth's axis, superimposed the precessional motion, caused by the plane of the Moon's orbit about the Earth rotating with respect to the ecliptic with a period of just under 19 years.

4. Planetary precessions due to perturbations from the other planets on the Earth's orbit.

III. THE CENTRALLY SYMMETRIC GRAVITATIONAL METRIC

Let us neglect the rotational motion of the Earth and consider that it is spherically symmetric and produces a gravitational field possessing central symmetry. The slight flatness at the poles is also neglected. Such a field can be produced by any centrally symmetric distribution of matter. Of course, not only the

distribution, but also the motion of the matter must be centrally symmetric. Let the center of mass of the Earth be located at the origin of the space-fixed reference system described in the previous section. Further hypothesize that the effects of all other heavenly bodies (Sun, Moon, planets, etc.) cannot be detected by the measuring accuracy of the atomic oscillators to be considered.

The gravitational metric (K. Schwarzschild, 1916), to terms of first order in v^2/c^2, is

$$ds^2 = ds_0^2 + \frac{2\phi}{c^2}(c^2dt^2 + dr^2), \tag{1}$$

where ds is the infinitesimal interval between two events infinitesimally close to one another, and ds_0 is the infinitesimal interval between two infinitesimally close events in the absence of a centrally symmetric gravitational field, i.e., the Galilean metric,

$$ds_0^2 = c^2dt^2 - (dx^2 + dy^2 + dz^2). \tag{2}$$

Furthermore,

$$dr^2 = \frac{(\overline{r} \cdot d\overline{r})^2}{r^2}, \tag{3}$$

where

$$\overline{r} = \hat{i}x + \hat{j}y + \hat{k}z, \tag{4}$$

with $\hat{i}$, $\hat{j}$, and $\hat{k}$ being the normal unit vectors directed along the x, y, and z axes, respectively, and

$$r = |\overline{r}|. \tag{5}$$

Finally,

$$\phi = -\frac{GM}{r}, \tag{6}$$

the gravitational potential. G is the universal gravitational constant and M is the mass of the central body, the Earth in this case. Further, assume that an order of accuracy of v^2/c^2 is sufficient for the experiment to be performed. The speed of the matter acted upon by the field is considered to be v. Of course, c is the speed of propagation of an electromagnetic signal in vacuum.

Schwarzschild found this metric from the Einstein equation of the gravitational field, which is the basic equation of general relativity theory. This equation relates the curvature properties of space-time to the energy momentum tensor of matter and field. In general relativity one further holds that the square of an interval appears as a quadratic form of general type in the coordinate differentials, i.e.,

$$ds^2 = g_{ik}\, dx^i\, dx^k , \tag{7}$$

where i and k run from 1 to 4 and the Einstein summation convention is used. For i and k running from 1 to 3, reference is made only to spatial coordinates. When i and/or k is 4, reference is made to the time coordinate $x^4 = ct$. The g_{ik} are the components of the metric tensor of the space-time metric and are always symmetric in the indices i and k, i.e.,

$$g_{ik} = g_{ki} . \tag{8}$$

Clearly, there are only 10 different quantities of g_{ik}, in general. In this paper the space-time metric is used rather than the spatial metric. Hence, in using Eq. (7), we will always sum i and k from 1 to 4. Note that for the Galilean or inertial metric of Eq. (2),

Cartesian space coordinates $x^{1,2,3} = x, y, z$, the time coordinate $x^4 = ct$, and the quantities g_{ik} are

$$g_{11} = g_{22} = g_{33} = -1, \; g_{44} = 1, \; g_{ik} = 0 \text{ for } i \neq k.$$

It is a fundamental assumption of physics that the interval squared, ds^2, is an invariant under any type of coordinate system transformation.

IV. EFFECTS OF A UNIFORMLY ROTATING COORDINATE SYSTEM ON THE GRAVITATIONAL METRIC

A prime system of coordinates that rotates uniformly about the z-axis with angular speed of rotation ω is now introduced (Figure 1).

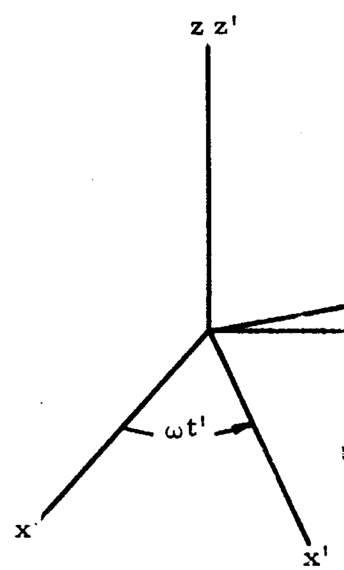

FIGURE 1. UNIFORMLY ROTATING COORDINATE SYSTEM

The transformation equations are given by

$$x = x'(\cos \omega t') - y'(\sin \omega t'), \tag{9}$$

$$y = x'(\sin \omega t') + y'(\cos \omega t'), \tag{10}$$

$$z = z', \tag{11}$$

$$t = t'. \tag{12}$$

Now, write the inverse transformation as

$$x' = x(\cos \omega t) + y(\sin \omega t), \tag{13}$$

$$y' = -x(\sin \omega t) + y(\cos \omega t), \tag{14}$$

$$z' = z \tag{15}$$

$$t' = t. \tag{16}$$

Using the above coordinate transformation the interval squared, ds^2, of Eq. (1) becomes

$$ds^2 = (ds_0')^2 + 2\omega y'dx'dt' - 2\omega x'dy'dt' + \frac{2\phi'}{c^2}(dr')^2 + \left[\frac{2\phi'}{c^2} - \frac{\omega^2(R')^2}{c^2}\right] c^2(dt')^2 , \tag{17}$$

where

$$(R')^2 \equiv (x')^2 + (y')^2 . \tag{18}$$

It is easily shown that

$$r' = r = |\overline{r}| = |\overline{r}'| ; \tag{19}$$

hence,

$$\phi' = \phi . \tag{20}$$

Also,

$$(dr')^2 = \frac{(\overline{r}' \cdot d\overline{r}')^2}{(r')^2} , \tag{21}$$

where

$$\overline{r}' = \hat{i}x' + \hat{j}y' + \hat{k}z' , \tag{22}$$

and

$$(ds_0')^2 = c^2(dt')^2 - [(dx')^2 + (dy')^2 + (dz')^2]. \tag{23}$$

In making this transformation it is assumed that it is not necessary to revert to the gravitational field equations and calculate the gravitational metric for the particular rotating coordinate system used. To be rigorously correct, one should start from the field equations and calculate the metric for the uniformly rotating coordinate system. However, it can be shown that any deviations are of much higher order than v^2/c^2, and out of range of the accuracy of the atomic oscillators. It is further assumed that the contribution of the Earth's spin (J. Lense and H. Thirring, 1918) to the gravitational field metric is also out of the range of accuracy of the atomic oscillators. No problems arise concerning the synchronization of clocks.

Next, find the relationship between the proper time (the time read by a clock moving with a given object), which from now on shall be denoted by τ', and the coordinate $x^{4'}$. This is done by considering two infinitesimally separated events occurring at the same point in space. Then, the interval ds between the two events is just $cd\tau'$, where $d\tau'$ is the proper time interval between the two events. The general expression (7) can be written as

$$(ds)^2 = g_{ik}dx^i dx^k = g'_{ik}dx^{i'}dx^{k'} . \tag{24}$$

So setting $dx^{1'} = dx^{2'} = dx^{3'} = 0$,

$$(ds)^2 = c^2(d\,\tau')^2 = g'_{44}(dx^{4'})^2 , \tag{25}$$

from which

$$d\,\tau' = \frac{1}{c}\sqrt{g'_{44}}\,dx^{4'} . \tag{26}$$

Now, suppose one wishes to count a certain number of events, N', in the prime system - say, the number of ticks of an atomic

oscillator - at a particular point. The number of events per unit proper time is then defined as f', and is given by

$$f' = \frac{dN'}{d\tau'} = \frac{dN'}{dx^{4'}}\,\frac{dx^{4'}}{d\tau'} \; , \tag{27}$$

or

$$f' = \frac{1}{c}\,\frac{dN'}{dt'}\left(\frac{c}{\sqrt{g'_{44}}}\right) , \tag{28}$$

since

$$dx^{4'} = cdt', \quad \frac{dx^{4'}}{d\tau'} = \frac{c}{\sqrt{g'_{44}}} \; . \tag{29}$$

Now, define

$$f'_0 = \frac{dN'}{dt'} \; , \tag{30}$$

the fundamental frequency of an event expressed in terms of the world time, $x^{4'}/c$. An examination of Eqs. (17) and (23) reveals that

$$\sqrt{g'_{44}} = \left(1 - \frac{\omega^2(R')^2}{c^2} + \frac{2\phi'}{c^2}\right)^{\frac{1}{2}} . \tag{31}$$

So, using Eqs. (30) and (31), one writes for Eq. (28)

$$f' = f'_0\left(1 - \frac{\omega^2(R')^2}{c^2} + \frac{2\phi'}{c^2}\right)^{-\frac{1}{2}} . \tag{32}$$

Expanding to first order gives

$$f' = f'_0\left(1 + \frac{\omega^2(R')^2}{2c^2} - \frac{\phi'}{c^2}\right). \tag{33}$$

For the time being, consider two identical atomic oscillators, which tick with the same fundamental frequency, f'_0 , located at positions 2 and 1 in the prime coordinate system. Then, by Eq. (33),

their frequencies are

$$f_2' = f_0'\left[1 + \frac{\omega^2 (R_2')^2}{2c^2} - \frac{\phi_2'}{c^2}\right], \tag{34}$$

$$f_1' = f_0'\left[1 + \frac{\omega^2 (R_1')^2}{2c^2} - \frac{\phi_1'}{c^2}\right], \tag{35}$$

from which one can easily construct

$$\frac{\Delta f'}{f_0'} = \frac{f_2' - f_1'}{f_0'} = \frac{\omega^2}{2c^2}\left[(R_2')^2 - (R_1')^2\right] + \frac{\phi_1' - \phi_2'}{c^2}. \tag{36}$$

Now, define

$$X \equiv \left| \frac{\omega^2}{2c^2}\left[(R_2')^2 - (R_1')^2\right] + \frac{\phi_1' - \phi_2'}{c^2} \right| . \tag{37}$$

Thus, one sees that the proper time is effected by making measurements in a uniformly rotating coordinate system superimposed upon the gravitational field. If it were possible to make the measurements at fixed points in the space-fixed system, then, obviously, the dimensionless quantity X would just be $X \equiv \left|\frac{\phi_1' - \phi_2'}{c^2}\right|$. The effect of a uniformly rotating coordinate system varies as the square of the position vector in the x'y' plane.

V. THE DOUBLE DOPPLER ELIMINATION METHOD

Let us suppose that we are going to compare the signals of two identical atomic oscillators: one of the oscillators will be at a fixed point in the uniformly rotating coordinate system at a tracking station on the Earth's surface close to the equator; the other oscillator will be almost fixed at a point in the uniformly rotating

coordinate system in synchronous orbit above the tracking station. The oscillator in synchronous orbit will not, in general, remain at a fixed point in the uniformly rotating coordinate system because of its orbit inclination angle and eccentricity. It is easily observed that if the orbit inclination and eccentricity are both zero, then the synchronous orbit satellite would remain at a fixed point in the uniformly rotating coordinate system.

Now, suppose that one translates (statically) to the tracking station, the origin of the uniformly rotating coordinate system at the Earth's center of mass. One still maintains a uniformly rotating coordinate system at the center of mass of the Earth and denotes the displaced coordinate system at the tracking station by the axes x_t, y_t, z_t. The axes of the coordinate system at the tracking station remain parallel and fixed in position to the x', y', z' axes in the uniformly rotating coordinate system. Also, locate an origin of coordinates at the synchronous orbit satellite and denote the axes by x_s, y_s, z_s. These axes are also required to remain parallel to the x', y', z' axes of the rotating coordinate system. However, the origin of coordinates at the synchronous orbit satellite will, in general, move with some relative velocity, $\overline{v}_s$, in the rotating coordinate system, and, hence, with the same relative velocity, $\overline{v}_s$, in the coordinate system at the tracking station. The only situation in which the synchronous orbit satellite has no relative velocity to the tracking station occurs at both zero eccentricity and orbit inclination. Figure 2 illustrates the motion of a synchronous orbit satellite characterized by some orbit inclination and

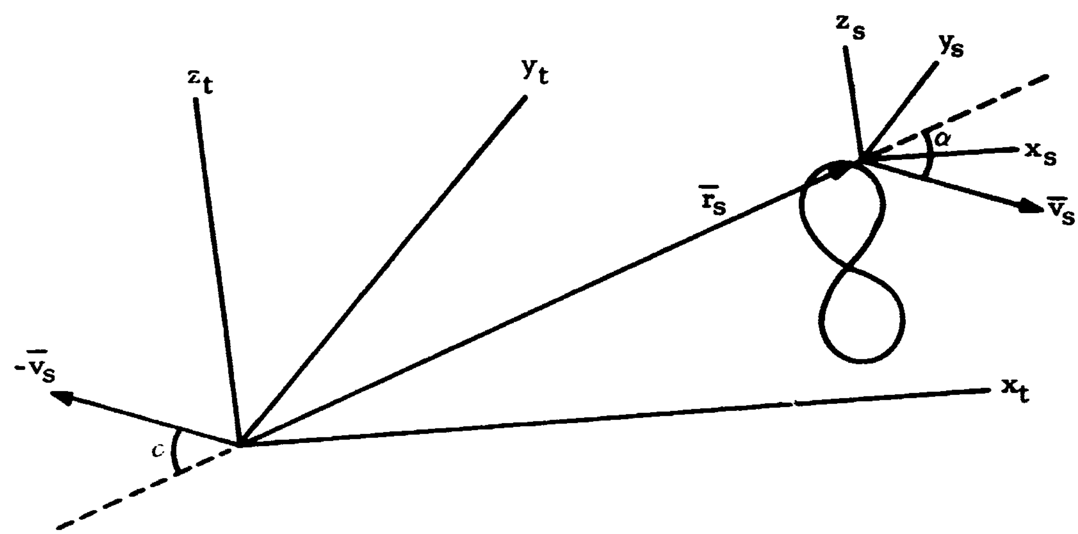

FIGURE 2. RELATIVE COORDINATE SYSTEMS OF TRACKING STATION AND SATELLITE

eccentricity. Note that if the position vector is reversed the tracking station appears to move with a velocity $-\overline{v}_s$ relative to the satellite. We denote the angle between the position vector, $\overline{r}_s$, of the satellite and the velocity vector, $\overline{v}_s$, of the satellite as α, and α is calculated in the usual way from

$$\cos \alpha = \frac{\overline{r}_s \cdot \overline{v}_s}{|\overline{r}_s|\,|\overline{v}_s|} \; . \tag{38}$$

Also note that the motion of a nonzero orbit inclination and nonzero eccentricity synchronous orbit satellite appears to be a "figure eight" to a ground observer at the tracking station. This "figure

eight" would shrink to zero for zero orbit inclination and zero eccentricity. The relative size of the "figure eight" depends on the magnitudes of the orbit inclination and eccentricity.

Now that the effects of proper time contraction and expansion in a gravitational field and a rotating coordinate system have been calculated, there remains the task of accounting for Doppler and special relativity effects. Also, the effect of electromagnetic propagation through a nonuniformly dense atmosphere shall be considered. To consider the Doppler and special relativity effects the assumption is made that the coordinate systems at the tracking station and synchronous orbit satellite are "inertial" to one another (i.e., that they do not undergo any accelerated motion relative to one another that can be detected by the accuracy of the identical atomic oscillators). In reality, they do not maintain a constant velocity relative to one another, but it is practical for the purposes of this report to assume that they do (for it is thought that the effects thereof are too small to be detected). Now, one proceeds to calculate the Doppler and special relativity effects on this basis, with, of course, the proper considerations given to the gravitational field and uniformly rotating coordinate system (calculated in Eq. 37). Finally, it is assumed that the fractional effect of the propagation of an electromagnetic wave through a nonuniformly dense medium can be lumped into a factor designated by P, which may be a function of several atmospheric variables. However, P is considered to be small enough that terms of the order of P^2 can be neglected. Now, define

$$\beta \equiv |\bar{v}_S| / c \,. \tag{39}$$

First of all, consider the fundamental frequency, f'_0. of the atomic oscillator at the tracking station to be broadcast to the synchronous orbit satellite. The frequency received or observed at the satellite is

$$f'_s = f'_0 (1 - \chi)(1 - P)(1 - \beta \cos \alpha)(1 - \beta^2)^{\frac{1}{2}}, \qquad (40)$$

where $(1 - \chi)$ represents the decrease in the observed frequency by virtue of its passage through an accelerated force field; $(1 - P)$ represents a decrease in the observed frequency by virtue of its passage through the nonuniformly dense atmosphere; $(1 - \beta \cos \alpha)$ accounts for the Doppler shift; and $(1 - \beta^2)^{\frac{1}{2}}$ represents the time dilatation factor from special relativity.* The terms χ, P, β^2, and $\beta^2 \cos^2 \alpha$ will be taken to be of the same relative order of magnitude, and any higher powers of these terms will be neglected. Hence, Eq. (40) is written as

$$f'_s = f'_0 \left(1 - \beta \cos \alpha - \frac{1}{2} \beta^2 - P - \chi\right), \qquad (41)$$

to the order of accuracy of this experiment.

Now, suppose that the frequency, f'_s, received at the synchronous orbit satellite, is rebroadcast and observed or received at the tracking station as f''. Taking into account the various effects, one writes

$$f'' = f'_s \frac{(1 + \chi)(1 - P)(1 - \beta^2)^{\frac{1}{2}}}{(1 + \beta \cos \alpha)}, \qquad (42)$$

and, expanding,

$$f'' = f'_s \left(1 - \beta \cos \alpha + \beta^2 \cos^2 \alpha - \frac{1}{2} \beta^2 - P + \chi\right). \qquad (43)$$

*Refer to appendices A, B, and C for the derivation of the effects due to the Doppler shift and time dilatation.

Substituting from Eq. (41) for f'_s into Eq. (43) yields (to the order of accuracy of our experiment)

$$f'' = f'_0 (1 - 2\beta \cos\alpha + 2\beta^2 \cos^2\alpha - \beta^2 - 2P). \tag{44}$$

Note that in Eq. (44) the "round trip" made by the frequency, f'_0, does not contain the factor χ, which accounts for the gravitational and rotational effects.

Now suppose that the fundamental atomic oscillator frequency, f'_0, is transmitted with the rebroadcasted signal, f'_s, from the synchronous orbit satellite. Let the frequency received on the ground from this signal be designated by f_r. One writes for f_r (considering all the previous factors),

$$f_r = f'_0 \frac{(1 + \chi)(1 - P)(1 - \beta^2)^{\frac{1}{2}}}{(1 + \beta \cos\alpha)}, \tag{45}$$

and expanding

$$f_r = f'_0 (1 - \beta \cos\alpha + \beta^2 \cos^2\alpha - \frac{1}{2}\beta^2 - P + \chi). \tag{46}$$

We can feed the frequency, f'', into a device at the tracking station that divides its phase by a factor of 2. After this is done, the resulting frequency becomes (by utilizing expression 44)

$$f = f'_0 (1 - \beta \cos\alpha + \beta^2 \cos^2\alpha - \frac{1}{2}\beta^2 - P). \tag{47}$$

Comparing f_r and f by beating them against one another gives

$$f_r - f \equiv \Delta f = f'_0 \chi, \tag{48}$$

where χ, as given by Eq. (37), contains only the effects of transmitting a frequency through a gravitational field in a uniformly rotating coordinate system. Considering the 21-centimeter frequency line of hydrogen and calculating χ for a synchronous

circular orbit with no inclination angle, one finds Δf to be in the neighborhood of 0.75 cycles per second. The present capability in electronic equipment makes this measurement possible to at least an order of magnitude better than previous gravitational shift type experiments. Pound and Snider* reported the effects of gravity on gamma radiation to a measured accuracy of slightly better than 1% (normalized result of 0.9990 ± 0.0076). They measured the recoil-free resonant absorption of the 14.4-keV gamma ray in Fe^{57} that traveled over a 75-foot vertical path in the Jefferson Laboratory at Harvard University. This was an improved version of an earlier experiment by Pound and Rebka in which an accuracy of only 10% was achieved.

It should be noted that any effects that may arise due to aberration on the angle, and uncertainties in position and velocity, have been assumed or hypothesized to be eliminated by the double Doppler elimination technique.

Finally, a brief discussion should be given concerning the factor P (which accounts for atmospheric conditions through which the electromagnetic signal is propagated). It is hypothesized that the atmospheric variations will be negligible compared to the 0.2 of a second round-trip time of a wavefront from the tracking station transmitter. But, suppose in the event of nonreciprocal propagation factors (such as a consistent upward thermal air flow, common in the tropics), the averaging of the propagation velocity over a

*Physical Review, volume 140, November 8, 1965, pages B788-B803.

two-way path is not the same as the propagation velocity over the one-way path from the satellite to the Earth. If such effects are significant they can be measured and accounted for by first broadcasting the signal, f'_0, from the satellite to the tracking station and then rebroadcasting the received signal back to the satellite. After the phase of this signal received at the satellite is divided by 2, it can be compared with the frequency received from the atomic oscillator at the ground tracking station. The result should be just the negative of the result of Eq. (48). Any absolute difference in these results would account for any nonreciprocal propagation factors in the nonuniformly dense atmosphere.

The factor X depends only on the position of the tracking station and the position of the synchronous orbit satellite in the uniformly rotating coordinate system. It would be interesting to have the equipment at the tracking station preprogrammed to calculate Δf in real time by using tracking data and compare these data on a real-time chart with the beating of the two signals together.

ACKNOWLEDGEMENTS

The author wishes to express his thanks and consideration to Dr. Russell D. Shelton and Dr. Nat Edmonson, Jr., for their encouragement and many fruitful discussions. This paper would not have been possible without their help and assistance.

APPENDIX A

TIME DILATATION FACTOR OF SPECIAL RELATIVITY

Consider the two inertial frames of reference of Figure 2 to be rotated statically through the same angles such that the velocity of the satellite system is directed along the x-axis Designate the rotated coordinate systems by the subscript r (illustrated in Fig. A-1. The purpose of this rotation is to make the calculation of time dilatation more easily understood by use of the standard Lorentz transformation. The resulting time dilatation factor is independent of the rotation.

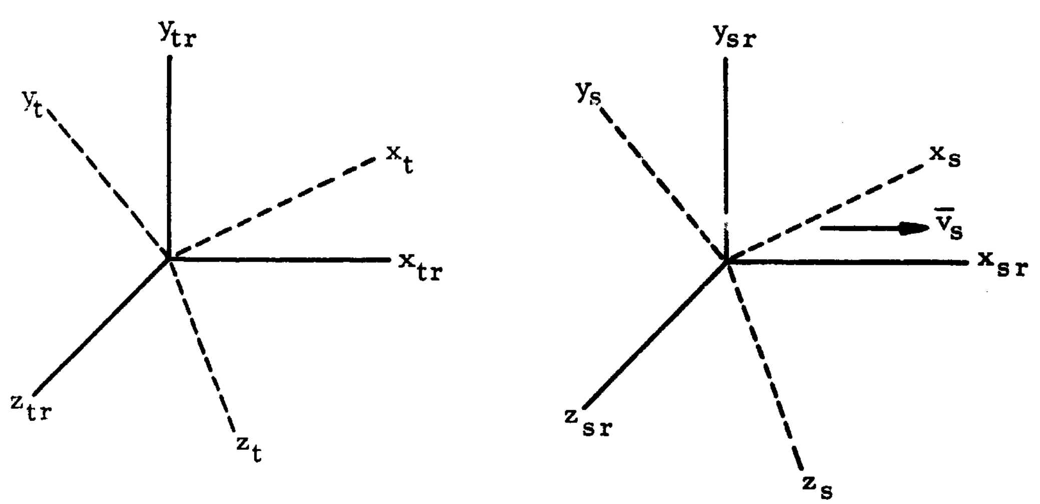

FIGURE A-1. ROTATED REFERENCE FRAMES OF TRACKING STATION AND SATELLITE

If an event is described in the tr system, then the description of the event in the sr system can be found by the well known Lorentz transformation,

$$x_{sr} = \gamma (x_{tr} - \beta ct_{tr}), \tag{A-1}$$

$$y_{sr} = y_{tr} , \tag{A-2}$$

$$z_{sr} = z_{tr} , \tag{A-3}$$

$$t_{sr} = \gamma (t_{tr} - \beta x_{tr}/c), \tag{A-4}$$

where

$$\gamma = (1 - \beta^2)^{-\frac{1}{2}}, \tag{A-5}$$

and β is defined by Eq. (39). On the other hand, if an event is described in the sr system, then the description of the event in the tr system can be found by the inverse transformation:

$$x_{tr} = \gamma (x_{sr} + \beta ct_{sr}), \tag{A-6}$$

$$y_{tr} = y_{sr} , \tag{A-7}$$

$$z_{tr} = z_{sr} , \tag{A-8}$$

$$t_{tr} = \gamma(t_{sr} + \beta x_{sr}/c). \tag{A-9}$$

Now, suppose that a clock is located at the origin of the sr system; if t_{sr_1} is the time of the start of one of its "ticks," then the observation time of this event in the tr system is (by Eq. A-9)

$$t_{tr_1} = \gamma t_{sr_1} , \tag{A-10}$$

since at the origin $x_{sr_1} = 0$. Let t_{sr_2} be the end of the "tick;"

then, similarly, we have

$$t_{tr_2} = \gamma t_{sr_2} , \qquad \text{(A-11)}$$

and

$$t_{tr_2} - t_{tr_1} = \gamma (t_{sr_2} - t_{sr_1}). \qquad \text{(A-12)}$$

The frequency of the "tick" in the sr system is then defined to be

$$\Omega_{sr} = \frac{1}{(t_{sr_2} - t_{sr_1})} , \qquad \text{(A-13)}$$

and the <u>observed</u> frequency of the "tick" in the tr system to be

$$\Omega_{tro} = \frac{1}{(t_{tr_2} - t_{tr_1})} . \qquad \text{(A-14)}$$

Then, by inversion of Eq. (A-12) we write

$$\Omega_{tro} = \frac{\Omega_{sr}}{\gamma} = \Omega_{sr}(1 - \beta^2)^{\frac{1}{2}}. \qquad \text{(A-15)}$$

Now, on the other hand, suppose that an <u>identical</u> clock is located at the origin of the tr system; if t'_{tr_1} is the time of start of one of its "ticks," then the observation time of this event in the sr system is (by Eq. A-4)

$$t'_{sr_1} = \gamma t'_{tr_1} , \qquad \text{(A-16)}$$

since, at the origin, $x'_{tr_1} = 0$. Let t'_{tr_2} be the end of the "tick;" then, similarly, we have

$$t'_{sr_2} = \gamma t'_{tr_2} , \qquad \text{(A-17)}$$

and

$$t'_{sr_2} - t'_{sr_1} = \gamma (t'_{tr_2} - t'_{tr_1}). \qquad \text{(A-18)}$$

Define the frequency of the "tick" in the tr system to be

$$\Omega_{tr} = \frac{1}{(t'_{tr2} - t'_{tr1})} , \qquad \text{(A-19)}$$

and the <u>observed</u> frequency of the "tick" in the sr system to be

$$\Omega_{sro} = \frac{1}{(t'_{sr2} - t'_{sr1})} . \qquad \text{(A-20)}$$

Then by inversion of Eq. (A-18), one writes

$$\Omega_{sro} = \frac{\Omega_{tr}}{\gamma} = \Omega_{tr}(1 - \beta^2)^{\frac{1}{2}} . \qquad \text{(A-21)}$$

Hence, it is concluded that the "tick" of a clock in an inertial frame moving with constant velocity $\overline{v}_s$ relative to an observer appears to be shorter to the observer than the "tick" of an identical clock in his own rest frame. Note that these results are independent of the orientation of the two inertial frames of reference, as long as the frames move with con. tant velocity relative to each other. By virtue of the clocks being defined to be <u>identical</u>, it is clear that

$$\Omega_{tr} \equiv \Omega_{sr} \qquad \text{(A-22)}$$

for the situation described.

APPENDIX B

THE DOPPLER EFFECT

CASE I: Source of Waves in Motion

When a source of waves is in motion through a stationary medium, the wavelength is changed. The waves sent out in the direction of motion of the source are shorter, and those in the opposite direction are longer than the waves from the source at rest. Consider a stationary source of wave motion. A wave emitted from this point after a period t_0' would have a wavelength λ'; but suppose, during one period of time t_0', the source moves with speed v_r in the direction of propagation of the waves. Then, the wavelength is shortened by a distance $v_r t_0'$, and the new wavelength is

$$\lambda = \lambda' - v_r t_0' . \tag{B-1}$$

If the waves travel with the velocity of light, c, then

$$\lambda = c t_0' - v_r t_0' = t_0'(c - v_r), \tag{B-2}$$

or

$$\lambda = \lambda' \left(\frac{c - v_r}{c}\right). \tag{B-3}$$

If, on the other hand, during one period of time t_0' the source moves with speed v_r in the direction opposite to the direction of propagation of the waves, the wavelength is lengthened by a distance $v_r t_0'$, and the new wavelength is

$$\lambda = \lambda' + v_r t_0' = \lambda' \left(\frac{c + v_r}{c}\right). \tag{B-4}$$

Equations (B-3) and (B-4) combine to form

$$\lambda = \lambda'\left(1 \pm \frac{v_r}{c}\right) , \tag{B-5}$$

the positive sign referring to the case in which the source is moving in the opposite direction from the waves and vice versa.

Suppose an observer locates the source of waves with a position vector $\bar{r}$. Further imagine that the source of waves travels with some arbitrary velocity $\bar{v}_s$. Let α be the angle between the position vector $\bar{r}$ and the velocity vector $\bar{v}_s$. Then compute $\cos\alpha$ by

$$\cos\alpha = \frac{\bar{r}\cdot\bar{v}_s}{|\bar{r}|\,|\bar{v}_s|} , \tag{B-6}$$

which enables one to write

$$\pm v_r = |\bar{v}_s| \cos\alpha . \tag{B-7}$$

With the aid of Eq. (B-7), we write for Eq. (B-5),

$$\lambda = \lambda' (1 + \beta\cos\alpha). \tag{B-8}$$

The velocity of the waves is not changed by the motion of the source. We define the frequency

$$\Omega' = \frac{c}{\lambda'} , \tag{B-9}$$

and write

$$\Omega = \frac{c}{\lambda} = \frac{c}{\lambda' (1 + \beta\cos\alpha)} , \tag{B-10}$$

or

$$\Omega = \frac{\Omega'}{1 + \beta\cos\alpha} , \tag{B-11}$$

for the new frequency received by an observer at rest with a source in motion with frequency Ω'.

CASE II: Observer of Waves in Motion

A change of frequency is encountered when the source is at rest, but the observer moves toward or away from the source. Let the source send out a train of waves such that Ω' is the frequency of the waves emitted from the source with some wave-length λ' related together by

$$\lambda' \Omega' = c, \tag{B-12}$$

where c is the velocity of light. Let an observer move toward the source with speed v_r. Then the observer sees

$$\lambda' \Omega = c + v_r. \tag{B-13}$$

If the observer moves away from the source with speed v_r then the observer sees

$$\lambda' \Omega = c - v_r . \tag{B-14}$$

Combining Eqs. (B-13) and (B-14) gives

$$\Omega = \frac{c \pm v_r}{\lambda'} = \Omega'\left(1 \pm \frac{v_r}{c}\right). \tag{B-15}$$

Similar to Case I, the observer is considered to be located with a position vector $\overline{r}$ and the observer moves with velocity $\overline{v}_s$. Then the same Eq. (B-6) results for cos α, and Eq. (B-7) can be utilized for $\pm v_r$. Hence for Eq. (B-15), one writes

$$\Omega = \Omega' (1 - \beta \cos \alpha), \tag{B-16}$$

as the new frequency received by an observer in motion with a source at rest with frequency Ω'.

APPENDIX C

EFFECTS OF BOTH TIME DILATATION AND DOPPLER

By using the results of Eqs. (A-21) and Eq. (B-16) one writes the following expression for the combined effects of the time dilatation of special relativity and Doppler for the case where the observer is in motion with respect to a source at rest:

$$(1 - \beta \cos \alpha)(1 - \beta^2)^{\frac{1}{2}}. \tag{C-1}$$

One must multiply the frequency of the source at rest by this factor in order to compute the frequency received by the observer in motion due only to the effects of time dilatation and Doppler.

By using the results of Eqs. (A-15) and (B-11) we write the following expression for the combined effects of the time dilatation of special relativity and Doppler for the case where the source is in motion with respect to an observer at rest:

$$\frac{(1 - \beta^2)^{\frac{1}{2}}}{(1 + \beta \cos \alpha)} . \tag{C-2}$$

One must multiply the frequency of the source in motion by this factor in order to compute the frequency received by an observer at rest due only to the effects of time dilatation and Doppler.

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